Izvestiya, Atmospheric and Oceanic Physics, Vol. 34, No. 3, 1998, pp. 370–377. Translated from Izvestiya AN. Fizika Atmosfery i Okeana, Vol. 34, No. 3, 1998, pp. 413–421.

Two Mechanisms of Surface Wave Generation: Kelvin–Helmholtz and Miles Instabilities

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Received August 7, 1996; in final form, December 17, 1996

Abstract—The growth rates of Kelvin–Helmholtz and Miles instabilities in the case of wave generation by wind are compared quantitatively as a function of the thickness of the boundary layer in the air. For a wind velocity greater than the threshold of Kelvin–Helmholtz instability and in the limit of a thin boundary layer with a thickness smaller than the wavelength of a perturbation, a parameter region is found where the Kelvin–Helmholtz instability dominates over the Miles instability. It is shown that the Miles instability due to the viscosity of air dominates if the thickness of the boundary layer is either too large or extremely small.

1. INTRODUCTION

Sea wave generation by wind has been studied intensively since the classical investigations by Kelvin in the middle of the last century. At present, at least two main mechanisms of the onset of instabilities at the air–water interface are known. The first mechanism is associated with the Kelvin–Helmholtz instability or the instability of a tangential discontinuity at the interface between two ideal incompressible fluids (see, for example, [1]). Assume that the ratio $\epsilon = \rho_2/\rho_1$ of the densities of the upper ρ_2 and lower ρ_1 fluids is small. For example, $\epsilon = 1.24 \times 10^{-3} \ll 1$ in the case of air and water. Then the equation for the complex phase velocity c of surface waves can be written accurately to $O(\epsilon)$ in the form

$$k^2c^2 - \omega_k^2 = -\epsilon k^2 U_0^2, \qquad (1)$$

where U_0 is the wind velocity, which is assumed to be parallel to the perturbation vector $\mathbf{k}$;

$$\omega_k^2 = \frac{k}{1+\epsilon}[g(1-\epsilon)+\alpha k^2] \qquad (2)$$

is the dispersion relation for gravity–capillary waves in the absence of wind; g is the acceleration of gravity; and α is the surface tension. It follows from (1) that the Kelvin–Helmholtz instability occurs when the critical wind velocity

$$U_{\text{cr}} = \frac{1}{\sqrt{\epsilon}} U_{\min} \qquad (3)$$

is exceeded. This velocity is determined by the minimum phase velocity $U_{\min} = \min\{\omega_k/k\}$ of surface gravity–capillary waves in the absence of wind. When the threshold velocity is exceeded, i.e., when $U_0 > U_{\text{cr}}$, the maximum growth rate of instability lies in the transition gravity–capillary region of the spectrum with $k \sim k_0 = \sqrt{g/\alpha}$ ($k_0 \simeq 3.664$ cm^{-1} and $U_{\text{cr}} \simeq 660$ cm/s for air and water). The instability is aperiodic in character, because the real part of the phase velocity of surface waves vanishes accurately to $O(\epsilon)$ if $U_0 > U_{\text{cr}}$.

The second mechanism of wave generation by wind is related to the Miles instability, which is due to fluid viscosity [2–4]. Viscosity causes a boundary layer to form in the upper light fluid near the interface. This instability is due to a shear flow $U = U(y)$ in the boundary layer, where y is the vertical coordinate measured from the interface. It is assumed that the mean flow in the lower heavy fluid is negligibly small ($U(0) = 0$) and that the velocity of the upper fluid $U = U_0 = \text{const}$ beyond the boundary layer. The Miles instability occurs at wind velocities that are much smaller than U_{cr}. The equation for the phase velocity c takes the form [4]

$$k^2(c+2ik\nu_1)^2 - \omega_k^2 = \gamma_r + i\gamma_i, \qquad (4)$$

where ν_1 is the coefficient of viscosity of the lower fluid, and the real quantities γ_r and γ_i depend on the viscosity of the upper fluid and the phase velocity c. A comparison of equations (1) and (4) shows that, if $U_0 > U_{\text{cr}}$, $\text{Im}(c) \gg 2k\nu_1$, $|\gamma_r + \epsilon k^2 U_0^2|$, $|\gamma_i| \ll \epsilon k^2 U_{\text{cr}}^2$, and $k \sim k_0$ (whence it follows that $\text{Re}(c) \ll \text{Im}(c)$), then the Kelvin–Helmholtz instability dominates; otherwise, the Miles instability dominates. It should be emphasized that, in this work, the thickness of the boundary layer is assumed to be smaller than or on the order of k_0^{-1}; i.e., this thickness is smaller than 1 cm in the case of air and water.

We notice that the Phillips mechanism [5] plays an important role in wave generation by wind in the gravitation region of the spectrum. This mechanism lies in the resonance generation of surface waves by turbulent fluctuations of pressure in the upper fluid. However, we restrict ourselves to perturbations on scales $k \sim k_0 = \sqrt{g/\alpha}$, when the Phillips mechanism may be disregarded.

Within the framework of the Kelvin–Helmholtz approach, a nonlinear stage of the onset of instability can be investigated. On the basis of perturbation theory, a nonlinear theory of Kelvin–Helmholtz instability was developed in [6] for a model of an ideal fluid. The characteristic slope of the interface was used as a small parameter. It was shown that nonlinearity in the first nonvanishing order in the amplitude of oscillations does not stabilize instability and leads to blast oscillation growth, which can be responsible for foam formation on the sea surface. This inference is in good agreement with observations, according to which a sharp increase in the fraction of the sea surface occupied by foam occurs at wind velocities $U \simeq 6$ m/s [7–10]. At the same time, unlike the Kelvin–Helmholtz instability, the Miles instability does not have a threshold at $U \sim 6$ m/s and grows continuously with wind velocity in this region of velocities. Nonlinear generalizations of the Miles theory (see, for example, [11, 12]) also do not predict a threshold at $U \sim 6$ m/s. In this connection, a comparison between the growth rates of Kelvin–Helmholtz and Miles instabilities at $U \sim U_{cr}$, which is the aim of this study, is of great importance. In particular, it is shown that the Kelvin–Helmholtz mechanism dominates over the Miles mechanism as the supercriticality $\delta = (U_0^2 - U_{cr}^2)/U_{cr}^2$ increases; and a sufficient value of supercriticality at which this domination occurs depends considerably on the thickness of the boundary layer h and the viscosity of the upper fluid (air).

The plan of this paper is as follows. Section 2 formulates a boundary problem for an inhomogeneous Orr–Sommerfeld equation, which is used to derive an equation of type (4) for the complex phase velocity c of surface waves. In Section 3, this boundary problem is solved for a specific profile of wind velocity $U(y) = U_0(1 - e^{-y/h})$ under an additional condition $khR \gg 1$, where $R = (U_0 h)/\nu_2$ is the Reynolds number and ν_2 is the kinematic viscosity of the upper fluid. It is shown that the Miles instability dominates in the case of $kh \sim 1$. In the next section, the solutions obtained in Section 3 are found to be greatly simplified in the limit of a thin (compared to the wavelength) boundary layer, $R^{-1} \ll kh \ll 1$. It is in this case that the domination of the Kelvin–Helmholtz mechanism turns out to be possible provided that an additional condition $kh(khR)^{1/3} \ll \delta$ is fulfilled. In the same section, the limit of a thin boundary layer, such that $khR \ll 1$ and $kh \ll 1$, is also discussed. It is shown that instability in this case depends only weakly on a specific profile of wind velocity $U(y)$, and the Kelvin–Helmholtz mechanism dominates under an additional condition $2/R \ll \delta$. In the final section, the results obtained are applied to an air flow over the water surface. It is indicated that, in real physical situations, the domination of Kelvin–Helmholtz instability is most probable when the air flows around the crests of steep sea waves, where the local wind velocity increases. The results of numerical simulation of the dependence $c(k)$ in the region $khR \sim 1$, where analytic results cannot be obtained, are also given in this section. In particular, the thickness of the boundary layer is found at which the growth rate of Kelvin–Helmholtz instability is maximum with respect to the Miles growth rate when the other parameters of the system are fixed.

2. BASIC EQUATIONS

Let us consider the flow of two viscous incompressible fluids whose unperturbed interface is horizontal. We will assume that the flow in the boundary layer of the upper light fluid is turbulent; however, the Reynolds number is considered to be large but finite, so that the viscosity of this fluid must be taken into account explicitly. Following [2–4], we will consider the so-called quasi-laminar model, in which the flow of the upper fluid is assumed to be plane-parallel and is described by the horizontal and vertical coordinates x and y, respectively. The velocity profile $\mathbf{U} = (U(y), 0)$ of an unperturbed flow of the upper fluid is obtained by averaging the horizontal velocity of the actual turbulent motion of the unperturbed flow of this fluid over the coordinate transverse to the (x, y)-plane. We superimpose a harmonic perturbation with wave vector $\mathbf{k} = (k, 0)$ on the uniform flow with velocity profile $U(y)$. The coordinate of the perturbed fluid interface takes the form

$$y_0 = a(t)e^{ikx}, \quad a(t) = a_0 e^{-ikct}. \qquad (5)$$

Thus, in the quasi-laminar model, the presence of turbulent fluctuations in the velocity of the upper fluid is taken into account in choosing the profile of wind velocity and is disregarded in considering small oscillations of this flow.

We introduce dimensionless variables in which both the characteristic thickness h of the boundary layer in the upper fluid near the interface and the asymptotic value of the unperturbed velocity of the upper light fluid $U(y)|_{y\to\infty} = U_0$ are equal to unity. We neglect the mean flow in the lower heavy fluid; i.e., $U(y) \equiv 0$ for $y \le y_0$.

For describing perturbations (5) against the background shear flow, Benjamin [13] introduced the curvilinear coordinates

$$\xi = x - iae^{ik(\xi + i\eta)}, \quad \eta = y - ae^{ik(\xi + i\eta)}, \qquad (6)$$

in which equation (5) has the form $\eta = 0$ accurate to the first-order terms with respect to ka. In this case, it is convenient to represent the stream function ψ as

$$\psi(\xi, \eta) = \int_0^{\eta}[U(\eta) - c]d\eta + [F(\eta) + [U(\eta) - c]e^{-k\eta}]ae^{ik\xi}. \qquad (7)$$

Then, linearization of the Navier–Stokes equations leads to the inhomogeneous Orr–Sommerfeld equation for $F(\eta)$

$$(U-c)(F''-k^2F)-U''F = \frac{1}{ikR}[F^{IV}-2k^2F''+k^4F+(U^{IV}-2kU''')e^{-k\eta}], \quad (8)$$

where R is the Reynolds number, which is equal to $R = U_0h/\nu_2$ if $\eta > 0$ and equal to $R = U_0h/\nu_1$ if $\eta < 0$; and ν_1 and ν_2 are the coefficients of kinematic viscosity of the lower and upper fluids, respectively.

In the linear approximation accurate to $O(k^2a^2)$, the kinematic condition at the fluid interface has the form $-ikcy_0 = -\left.\frac{\partial\psi}{\partial x}\right|_{y=y_0}$. Writing this relation in (ξ, η)-coordinates and using formula (7) for the stream function, we obtain in the linear approximation that

$$F(0) = c. \quad (9)$$

The continuity of the horizontal velocity at the interface determines the second kinematic condition. We represent this fluid velocity as

$$\left.\frac{\partial\psi}{\partial y}\right|_{y=y_0} = uy_0 - c, \quad (10)$$

where the quantity u is to be determined. In the curvilinear (ξ, η)-coordinates, it follows from (7) and (10) that

$$F'(0) = -U'(0)+u. \quad (11)$$

In view of the initial assumption, there is no mean flow in the lower fluid ($U(\eta)|_{\eta<0} = 0$). As a result, the Orr–Sommerfeld equation (8) reduces to a homogeneous differential equation with constant coefficients whose general solution, decreasing as $\eta \longrightarrow -\infty$, has the form

$$F|_{\eta<0} = C_1e^{k\eta}+C_2e^{\eta\sqrt{k^2-ikc/\nu_1}}, \quad (12)$$

where the unknown constants C_1 and C_2 are determined from the kinematic conditions (9) and (11). The unknown quantities u and c can be found using the dynamic conditions imposed on the stress tensor σ at the interface (i.e., at $\eta = 0$):

$$\sigma_1^{xy} = \sigma_2^{xy}, \quad \sigma_1^{yy} = \sigma_2^{yy}-\rho_1\alpha k^2ae^{ik\xi}, \quad (13)$$

where the indices 1 and 2 relate to the lower and upper fluids, respectively. We represent the components of the stress tensor for a harmonic perturbation in the upper fluid as

$$\sigma_2^{yy}|_{\eta=0} = -p_0-2ik\frac{F'(0)}{R}\rho_2ae^{ik\xi},$$
$$(p_0, \sigma_2^{xy})|_{\eta=0} = \rho_2(P,T)ae^{ik\xi}, \quad (14)$$

where p_0 is the normal pressure at $\eta = 0$. Henceforth, $R = U_0h/\nu_2$. Using formula (7) and the definition for σ given in [1], we linearize the horizontal component of the Navier–Stokes equation. Switching next to the curvilinear coordinates, we obtain the following relations for P and T (see [4, 13]):

$$P = k^2\int_0^\infty(U-c)Fd\eta-\frac{i}{kR}\times\left[k^2F'(0)+\int_0^\infty(k^4F-k^2U''e^{-k\eta})d\eta\right], \quad (15)$$
$$T = \frac{1}{R}[F''(0)+k^2c+U''(0)].$$

The linearization of the vertical component of the Navier–Stokes equation also yields the equivalent relation for P:

$$P = U'F-(U-c)F'-\frac{i}{kR}\times[F'''-k^2F'+(U'''-kU'')e^{-k\eta}]|_{\eta=0}. \quad (16)$$

The stress tensor in the lower fluid is obtained by substituting ν_1 for ν_2 and $\rho_1 = 1/\epsilon$ for $\rho_2 \equiv 1$ and by taking into account that $U(\eta) \equiv 0$ for $\eta < 0$.

We will assume that $|c|/k\nu_1 \gg 1$. From (9) and (11)–(14), we find the boundary conditions for the Orr–Sommerfeld equation (8) in the upper fluid

$$F(0) = c, \quad F'(0) = -U'(0)+kc, \quad (17)$$

as well as the equation of type (4) for the complex phase velocity c of surface waves [4]

$$(c+2ik\nu_1)^2-\frac{\omega_k^2}{k^2} = \frac{\epsilon}{k}\left(P+2ik\frac{F'(0)}{R}+iT\right),$$
$$\epsilon = \frac{\rho_2}{\rho_1} \ll 1, \quad (18)$$

where ω_k is given by formula (2).

Two other boundary conditions for the Orr–Sommerfeld equation (8) follow from the finiteness of the perturbation:

$$F, F'|_{\eta\to\infty} \longrightarrow 0. \quad (19)$$

Equation (8) subject to the boundary conditions (17) and (19) and equations (15) and (18) form a closed system of eigenvalue equations for the complex phase velocity c of surface waves.

3. SOLUTION TO THE EIGENVALUE PROBLEM IN THE CASE OF A SPECIFIC WIND VELOCITY PROFILE FOR $kR \gg 1$

As an approximation of the unperturbed wind velocity profile $U(\eta)$, it is convenient to take the function

$$U(\eta) = U_0(1-e^{-\eta/h}) \equiv 1-e^{-\eta}. \quad (20)$$

This choice of the velocity profile makes it possible to obtain an explicit solution for the inviscid part of the Orr–Sommerfeld equation (8)—the so-called Rayleigh equation

$$(U-c)(F''-k^2F)-U''F = 0. \quad (21)$$

On the other hand, the fact that the curvature of the velocity profile $U''(0) \neq 0$ is other than zero at $\eta = 0$ means that the complex velocity c varies only slightly under small variations of profile (20). Previously, specific profiles $U(\eta)$ that admitted explicit solutions to the Orr–Sommerfeld equation were also suggested [13]; however, the curvature $U''(0) = 0$ in those examples. This is a very special case, because $U''(0)$ appears explicitly in the asymptotic expansion of the solutions to the Orr–Sommerfeld equation (8) as $kR \longrightarrow \infty$ [15] and the vanishing of the curvature qualitatively changes these solutions.

Turning back to the velocity profile (20), we make the following substitution in the Rayleigh equation (21):

$$z = 1-e^{-(\eta-\eta_c)}, \quad \phi = (1-z)^{-k}F, \quad (22)$$

where the quantity η_c is defined by the condition $U(\eta_c) = c$; i.e., $\eta_c = -\ln(1-c)$. As a result, equation (21) reduces to the hypergeometric equation

$$(1-z)z\phi''-(2k+1)z\phi'+\phi = 0. \quad (23)$$

As $\eta \longrightarrow \infty$ and $z \longrightarrow 1$, the boundary conditions require that a solution to (23) be expressed through the standard hypergeometric function in the neighborhood of $z = 1$:

$$\phi = \alpha F(k+\sqrt{k^2+1}, k-\sqrt{k^2+1}, 2k+1, 1-z), \quad \alpha = \text{const}. \quad (24)$$

Function (24) is a linear combination of two hypergeometric functions of z, one of which is regular and the other has a logarithmic singularity as $\eta \longrightarrow \eta_c$ and $z \longrightarrow 0$ [14]. As a result, in the neighborhood of $\eta \longrightarrow \eta_c$ the solution of the Rayleigh equation (21) that satisfies the boundary conditions (19) can be written as

$$F_R = \frac{\alpha\Gamma(1+2k)}{\Gamma(k+\sqrt{k^2+1})\Gamma(k-\sqrt{k^2+1})} \times\{[\psi(k+\sqrt{k^2+1})+\psi(k-\sqrt{k^2+1})-\psi(2)] \times(\eta-\eta_c)-1+(\eta-\eta_c)\ln(\eta-\eta_c)+(\gamma-k)(\eta-\eta_c)\}, \quad (25)$$

where Γ and ψ are the standard gamma function and psi function and $\gamma = 0.5772...$ is the Euler constant.

The solutions of the Rayleigh equation (21) offer a close approximation of the solutions of the Orr–Sommerfeld equation (8) everywhere except in narrow layers in which the viscosity of the upper fluid is important. In the vicinity of the so-called coincidence layer, where $\eta \longrightarrow \eta_c$ and $U(\eta) \longrightarrow c$, resonance phenomena occur due to the coincidence between the velocity of the main flow $U(\eta)$ and the phase velocity of surface waves c. The viscous right-hand side of the Orr–Sommerfeld equation (8) becomes important. The thickness of the coincidence layer has the order $\mu = 1/(kR)^{1/3}$, and the solutions of (8) are expressed in powers of μ in the neighborhood of $\eta = 0$. In the general case, there is a viscous sublayer of thickness $1/(kR)^{1/2}$ in the neighborhood of $\eta = 0$, where the viscous terms of the Orr–Sommerfeld equation are also of importance. However, $\mathrm{Re}(c) \longrightarrow 0$ and $\eta_c \longrightarrow 0$ near the Kelvin–Helmholtz instability threshold; therefore, the viscous sublayer merges with the coincidence layer and all solutions are expanded in powers of the parameter μ [15]. In the zeroth order in μ, the general solution of the Orr–Sommerfeld equation in the coincidence layer has the form

$$F = \beta_1+\beta_2\theta+\beta_3\chi_3(\theta), \quad (26)$$

where $\theta = (\eta-\eta_c)/\mu$, $\beta_{1,2,3} = \text{const}$, $\chi_3(\theta) = \int_\infty^\theta d\theta'\int_\infty^{\theta'} d\theta''\sqrt{\theta''}H_{1/3}^{(1)}\left[\frac{2}{3}(i\theta'')^{3/2}\right]$, and $H_{1/3}^{(1)}$ is the Hankel function of the first kind. In the region $\theta \longrightarrow \infty$, the terms up to first order must be retained in the expansion in powers of μ. Taking into account that $\chi_3(\theta)|_{\theta\to\infty} \longrightarrow 0$, we obtain

$$F = \beta_1+\beta_2\theta+\beta_1\mu\frac{U''(\eta_c)}{U'(\eta_c)}\theta\ln\theta, \quad (27)$$

where we may set $U''(\eta_c)/U'(\eta_c) \simeq U''(0)/U'(0) = -1$, because η_c is small. The unknown coefficients β_1, β_2, β_3, and α are found from the conditions of sewing the outer (25) and inner (27) expansions and from the boundary conditions (17) at $\eta = 0$. In the limit $\eta_c \longrightarrow 0$, $\mu \longrightarrow 0$, and $|c| \ll \mu$, we obtain from (16) that

$$P = \frac{F'''(0)}{ikR} = \{-\Gamma(k+\sqrt{k^2+1})\Gamma(k-\sqrt{k^2+1})\}\Big/\Big\{\Gamma(1+2k)\Big[\psi(k+\sqrt{k^2+1})+\psi(k-\sqrt{k^2+1}) -\psi(2)+\gamma-k+\ln\mu+\frac{\chi_3'(0)}{\mu\chi_3(0)}\Big]\Big\}. \quad (28)$$

Since μ is small in view of the initial assumption, equation (28) is greatly simplified under the additional condition $k \sim 1$:

$$P = -\frac{\Gamma(k+\sqrt{k^2+1})\Gamma(k-\sqrt{k^2+1})}{\Gamma(1+2k)}\mu\frac{\chi_3(0)}{\chi_3'(0)}. \quad (29)$$

Thus, for k greater than or on the order of unity, the terms that depend explicitly on the coefficient of viscosity play a leading part. Therefore, the initial assumption $\eta_c \longrightarrow 0$ is violated, and the Miles instability dominates over the Kelvin–Helmholtz instability. Physically, this result is quite reasonable, because the dependence of the growth rate on the structure of the boundary layer is of importance when its thickness is greater than or on the order of the perturbation wavelength, and this layer can in no way be approximated by a tangential discontinuity as is done in the Kelvin–Helmholtz theory. Moreover, as seen from equation (28), even without considering the viscosity of the upper fluid, the growth rate in this case will be different from the Kelvin–Helmholtz growth rate.

In the conclusion of this section, we note that the possibility of reducing the Rayleigh equation to the hypergeometric equation for the velocity profile (20) enables us to determine c without the additional assumption $|c| \ll 1$ in the case when the coincidence layer and the viscous layer are separated asymptotically. However, this is beyond the scope of this study.

4. THIN BOUNDARY LAYER LIMIT

4.1. Limit of $k \ll 1$, $kR \gg 1$

If the thickness of the boundary layer h is sufficiently smaller than the wavelength; i.e., $k \ll 1$ in dimensionless variables but the condition $kR \gg 1$ still holds, then formulas (25) and (28) are greatly simplified. In particular, accurate to $O(k^2)$, we have

$$F_R = \alpha[k+(\eta-\eta_c)-k(\eta-\eta_c)\ln(\eta-\eta_c)]$$

$$P = -k\Big/\left(1-\frac{\chi_3'(0)k}{\chi_3(0)\mu}\right), \quad (30)$$

$$\frac{\chi_3'(0)}{\chi_3(0)} = -1.1153 - i0.6440\ldots$$

Similarly, it follows from (15) that the tangential stress

$$T = k^2\mu\frac{\chi_3''(0)}{\chi_3(0)}, \quad \frac{\chi_3''(0)}{\chi_3(0)} = 0.6858 + i1.1880\ldots \quad (31)$$

is on the order of $k^2\mu$ and, consequently, is negligibly small compared to P.

The Kelvin–Helmholtz theory states that $P + iT = -kU_0^2 \equiv -k$; therefore, if $k/\mu \sim 1$, the Miles instability obviously dominates. If $k/\mu \ll 1$, equations (18), (30), and (31) yield the following relation for the complex phase c at $k \simeq k_0 = \sqrt{g/\alpha}$:

$$k^2(c+2ik\nu_1)^2 = \epsilon k^2\left(-\delta+\frac{\chi_3'(0)k}{\chi_3(0)\mu}\right), \quad (32)$$

where $\delta = (U_0^2 - U_{cr}^2)/U_{cr}^2$ is the supercriticality, and the critical velocity U_{cr} is given by equation (3). According to (32), the Kelvin–Helmholtz instability dominates over the Miles instability under the following conditions:

$$\frac{k}{\mu} \ll 1, \quad \frac{k}{\mu} \ll \delta, \quad \mu \ll 1. \quad (33)$$

The Miles instability plays a leading part when $k/\mu \sim \delta$.

We notice that, in the case of a thin boundary layer ($k \ll 1$), an expansion of type (30) can be obtained from the approximate Heisenberg solutions of the Rayleigh equation (21) for an arbitrary velocity profile $U(\eta)$ [15].

4.2. Limit of $k \ll 1$, $kR \ll 1$

If the thickness of the boundary layer is so small that kR is smaller than or on the order of unity, the validity of asymptotic expansions in μ is violated. However, if the additional condition $kR \ll 1$ is fulfilled, it is possible to obtain the following analytic relation for P and T (see the Appendix):

$$P + 2ik\frac{F'(0)}{R} + iT = -k\left(1+\frac{2i}{R}\right) + O\left(k\sqrt{\frac{k}{R}}\right). \quad (34)$$

An analysis of (18) and (34), similar to that of equation (32) in Section 4.1, shows that, when

$$k \ll 1, \quad kR \ll 1, \quad 2/R \ll \delta, \quad (35)$$

the Kelvin–Helmholtz instability dominates over the Miles instability. In view of the proportionality between the Reynolds number R and the thickness of the boundary layer h, conditions (35) are valid for values of h that are not too small.

If conditions (33) and (35) are satisfied, the dependence of $\mathrm{Im}(c)$ on k will have a sharp peak at $k \sim k_0$, which is due to the Kelvin–Helmholtz instability. Beyond the neighborhood of k_0, the imaginary part of the phase velocity $\mathrm{Im}(c)$ is significantly smaller, and the real part is on the order of the phase velocity of gravity–capillary waves ω_k/k in the absence of wind.

We notice that, physically, the distinction between the Kelvin–Helmholtz and Miles instabilities lies in the fact that the former results from the pressure that arises during air flow around a wavy surface and is in phase with the elevation of the surface. Therefore, the growth rate of Kelvin–Helmholtz instability is positive if this pressure is greater than the damping action of the gravity and capillary forces. The latter—the Miles instability—is mainly due to the normal stress, which is in phase with the slope of the wavy surface and is greater in magnitude than dissipative forces. However, as indicated in the next section, the tangential stress also contributes significantly to the Miles instability in the intermediate case $k \ll 1$, $kR \sim 1$.

5. CONCLUDING REMARKS

In the specific case of air flow over the water surface, where $k \simeq k_0 = \sqrt{g/\alpha} = 3.664\ \text{cm}^{-1}$, $\nu_2 = 0.15\ \text{cm}^2/\text{s}$, and $U_{cr} = 660$ cm/s, the above results concerning the regions in which the Kelvin–Helmholtz instability dominates take the following form:

(a) for sufficiently thin boundary layers, it follows from (33) that $0.008\ \text{cm} \ll h \ll 0.27\ \text{cm}$, $\delta \gg (h/(6.6 \times 10^{-2}\ \text{cm}))^{5/3}$;

(b) in the limit of very thin boundary layers, it follows from (35) that $h \ll 0.008$ cm, $\delta \gg 4.5 \times 10^{-4}\ \text{cm}/h$.

In the intermediate region $k \ll 1$, $kR \sim 1$ between the limits (a) and (b), the thickness of the boundary layer h is on the order of the thickness of the coincidence layer, and asymptotic techniques cannot be applied to obtain analytical results. In the case of air and water, the complex phase velocity $c(k)$ at $k \sim k_0$ was calculated numerically in this region. The boundary problem (8), (17), (19) was solved by the orthogonal factorization method [15, 16] for each value of c, after which the roots of (18) were found by the modified Newton method. As a result, the thickness of the boundary layer was found to take on the value $h \longrightarrow 0.02/k_0 = 0.0054$ cm, at which the Kelvin–Helmholtz instability dominates at the smallest supercriticality $\delta = (U_0^2 - U_{cr}^2)/U_{cr}^2$. At this thickness of the boundary layer, $\text{Re}(c) \ll \text{Im}(c)$; i.e., the Kelvin–Helmholtz mechanism plays a crucial role for $\delta > 0.1$. In addition, the validity of the formulas obtained in Sections 3, 4.1, and 4.2 is completely confirmed by numerical calculations. We notice that the numerical calculations also show that, in the limit $k \ll 1$, $kR \sim 1$, the contribution of the tangential stress to the growth rate of Miles instability is generally on the same order as that of the normal stress, unlike the limits considered in Sections 3, 4.1, and 4.2, where the contribution of the normal stress is predominant.

Thus, on the basis of analytic and numerical results, we arrive at the conclusion that, for the Kelvin–Helmholtz instability to dominate over the Miles instability, the wind speed must be noticeably greater (by no less than 10%) than the critical value $U_{cr} = 660$ cm/s. At the same time, the thickness of the boundary layer must be neither too large, so that the perturbation wavelength is greater than h, nor too small, because an abrupt change in the velocity $U(\eta)$ causes the viscous terms to dominate in the growth rate of instability.

For real sea waves, the domination of Kelvin–Helmholtz instability is most probable both during abrupt gusts and near the crests of steep sea waves, because the air flow around the latter causes the local wind velocity to increase up to values higher than the threshold value U_{cr}.

We note that the nonlinear theory of Kelvin–Helmholtz instability developed in [6] used the smallness condition for the supercriticality δ. For this reason, the applicability of this theory to wave generation by wind is associated at least with rather stringent limitations on the thickness of the boundary layer h in the air near the water surface. Nevertheless, condition (33) is indicative of a wide region of validity of the Kelvin–Helmholtz theory in studies of interface instability either in the case where the kinematic viscosity of the upper fluid is significantly smaller than that of air or under noticeable changes in other parameters of the system under consideration, such as the surface tension α, the acceleration of gravity g, and the density ratio $\epsilon = \rho_2/\rho_1$.

ACKNOWLEDGMENTS

The author is grateful to E.A. Kuznetsov for useful discussions.

This work was supported by the Russian Foundation for Basic Research (project no. 97-01-00093); the Program of state support for the leading scientific schools (grant no. 96-15-96093); the Landau Scholarship Foundation; KFA, Forschungzentrum, Juelich, Germany; and the INTAS Foundation (grant no. 96-0954).

APPENDIX

Let us consider the limit of a small thickness of the boundary layer h such that $k \ll 1$ and $kR \ll 1$. In the region $\eta \gg 1$, the Orr–Sommerfeld equation (8) reduces to a homogeneous differential equation with constant coefficients. Its general solution is given by a sum of two exponential functions:

$$F = \alpha_1 e^{-k\eta} + \alpha_2 e^{-\lambda\eta}, \quad \alpha_{1,2} = \text{const}, \qquad \lambda = \sqrt{k^2 + ikR(1-c)}. \tag{36}$$

The applicability of (36) is violated in a narrow layer with the characteristic thickness $h \equiv 1$ near the interface. At the same time, solution (36) varies on the scales k^{-1} and $(kR)^{-1/2}$, which are much larger than h. The velocity profile $U(\eta)$ is closely approximated on these scales by a velocity jump from $U(0) = 0$ up to $U(0 + 0) = U_0$; and, as shown below, the quantities P and T depend only slightly on the specific form of the profile $U(\eta)$. In the limit $kR \ll 1$, if $\lambda\eta \ll 1$, the right-hand side dominates in the Orr–Sommerfeld equation and the latter reduces to an inhomogeneous differential equation with constant coefficients

$$\frac{1}{ikR}[F^{IV} - 2k^2F'' + k^4F + (U^{IV} - 2kU''')e^{-k\eta}] = 0. \tag{37}$$

Straightforward substitution demonstrates that its solution has the form

$$F = (C_1 + C_2\eta)e^{k\eta} + (C_3 + C_4\eta)e^{-k\eta} + e^{k\eta}\int_{\eta}^{\infty} e^{-2k\eta}U'(\eta)d\eta,$$

where $\lambda\eta \ll 1$ and $C_{1,2,3,4}$ are arbitrary constants. We will assume that the wind velocity is so much greater than the threshold of Kelvin–Helmholtz instability (3) that the phase velocity may be neglected ($c \longrightarrow 0$) in calculating the right-hand side of (18). In this case, $\lambda = \sqrt{k^2 + ikR}$. Expanding the homogeneous part of this solution in powers of η up to the first-order terms for $k\eta \ll 1$ and satisfying the boundary conditions (17), we obtain

$$F = -(1 + k\eta)\int_0^{\infty} e^{-2k\eta}U'(\eta)d\eta + e^{k\eta}\int_{\eta}^{\infty} e^{-2k\eta}U'(\eta)d\eta. \quad (38)$$

It is convenient to write the first integral in this formula as

$$-\int_0^{\infty} e^{-2k\eta}U'(\eta)d\eta = -1 + kB,$$
$$B = -2\int_0^{\infty} e^{-2k\eta}(U(\eta) - 1)d\eta, \quad (39)$$

where $B = O(1)$ if $U'(\eta)$ decreases as $\eta \longrightarrow \infty$ no slower than $1/\eta^2$, which is assumed to be fulfilled below. The regions of validity of solutions (36) and (38) overlap for $\eta \gg 1$, $\lambda\eta \ll 1$, and $k\eta \ll 1$. In this case, $B \longrightarrow 0$. Therefore, expanding (36) up to the first-order terms in η and equating (36) to (38), we obtain a solution of the Orr–Sommerfeld equation (8) that is uniformly suitable at any η:

$$F = \alpha_1 e^{-k\eta} + \alpha_2 e^{-\lambda\eta} + e^{k\eta}\int_{\eta}^{\infty} e^{-2k\eta}U'(\eta)d\eta, \quad (40)$$

where

$$\alpha_1 = -1 + kB + \frac{2k(-1 + kB)}{-k + \lambda},$$
$$\alpha_2 = -1 + kB - \alpha_1, \quad (41)$$

and the constant B is given by (39). Substituting (40) into (8), one can verify that corrections to solution (40) are on the order of k^2R. We notice that the Reynolds number R is proportional to the thickness of the boundary layer h. Therefore, it is convenient to use the quantity

$$\frac{R}{k} = \frac{U_0}{k\nu_2}, \quad (42)$$

which is independent of h, to estimate different terms in (4). We assume that $R/k \gg 1$ (specifically, $R/k \simeq R/k_0 = 1200$ for air and water). Under this condition, $\lambda \simeq \sqrt{ikR}(1 + O(k/R))$ and equations (41) are greatly simplified:

$$\alpha_1 = -1 - \frac{2k}{\sqrt{ikR}} + i\frac{2k}{R} + kB + O\left(k\sqrt{\frac{k}{R}}\right),$$
$$\alpha_2 = \frac{2k}{\sqrt{ikR}} - i\frac{2k}{R} + O\left(k\sqrt{\frac{k}{R}}\right). \quad (43)$$

From (15), (40), and (43), we find the leading-order (in k/R) corrections to the pressure $P_0 = -k$ in the Kelvin–Helmholtz theory:

$$P = -k\left(1 + \frac{2k}{\sqrt{ikR}}\right) + O\left(\frac{k^2}{R}\right),$$
$$T = \frac{i2k^2}{\sqrt{ikR}} + O\left(\frac{k^2}{R}\right). \quad (44)$$

At the same time, the viscous part of the normal stress tensor (14) $2ikF'(0)/R = -2ikU'(0)/R$ dominates over corrections (44) in view of $R/k \gg 1$ unless $U'(0)$ is an abnormally small quantity compared to U_0/h. To be more specific, we set $U'(0) = U_0/h = 1$ (this is the case for (20)). Then we obtain the final relation for the right-hand side of equation (18)

$$P + 2ik\frac{F'(0)}{R} + iT = -k\left(1 + \frac{2i}{R}\right) + O\left(k\sqrt{\frac{k}{R}}\right);$$

i.e., we arrive at equation (34), which was used in Section 4.2.

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Translated by Z.I. Feizulin